

# Conditions for the freezing phenomena of geometric measure of quantum discord for arbitrary two-qubit X states under non-dissipative dephasing noises

Wei Song,<sup>1</sup> Long-Bao Yu,<sup>1</sup> Da-Chuang Li,<sup>1,2</sup> Ping Dong,<sup>1</sup> Ming Yang,<sup>3</sup> and Zhuo-Liang Cao<sup>1</sup>

<sup>1</sup>*School of Electronic and Information Engineering, Hefei Normal University, Hefei 230061, China*

<sup>2</sup>*Hefei National Laboratory for Physical Sciences at the Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China*

<sup>3</sup>*School of Physics and Material Science, Anhui University, Hefei 230039, China*

(Dated: February 27, 2013)

We study the dynamics of geometric measure of quantum discord (GMQD) under the influences of two local phase damping noises. Consider the two qubits initially in arbitrary X states, we find the necessary and sufficient conditions for which GMQD are unaffected for a finite period. It is further shown that such results also hold for the non-Markovian dephasing process.

PACS numbers: 03.65.Ud, 03.67.Mn, 03.65.Yz

Quantum discord has received a lot of attentions due to its potential to serve as an important resource in the deterministic quantum computation with one pure qubit (DQC1) [1, 2]. It indicates that unentangled states may process quantumness which can be exploited in certain types of quantum information processing tasks. There are several versions of quantum discord. The first definition of quantum discord is introduced by Ollivier and Zurek [3] and, independently, by Henderson and Vedral [4]. The quantum discord of a composite system AB is defined by  $D_A \equiv \min_{\{E_i^A\}} \sum_i p_i S(\rho_{B|i}) + S(\rho_A) - S(\rho_{AB})$ ,

where  $S(\rho_{AB}) = -\text{Tr}(\rho_{AB} \log_2 \rho_{AB})$  is the von Neumann entropy and the minimum is taken over all positive operator valued measures (POVMs)  $\{E_i^A\}$  on the subsystem A with  $p_i = \text{Tr}(E_i^A \rho_{AB})$  being the probability of the i-th outcome and  $\rho_{B|i} = \text{Tr}_A(E_i^A \rho_{AB})/p_i$  being the conditional state of subsystem B. The minimum can also be taken over the von Neumann measurements. Below we term this definition as measurement based quantum discord. Because of the minimization taken over all possible POVM, or von Neumann measurements, is generally difficult to calculate measurement based discord. In order to overcome this difficulty geometric measure of quantum discord (GMQD) has been introduced by Dakic *et al* [5]. GMQD is defined by  $D_A^g = \min_{\chi \in \Omega_0} \|\rho - \chi\|^2$ , where  $\Omega_0$  denotes the set of

zero-discord states and  $\|X - Y\|^2 = \text{Tr}(X - Y)^2$  is the square norm in the Hilbert-Schmidt space. The subscript A denotes that the measurement is taken on the system A. Especially, Dakic *et al* [6] show that the GMQD is related to the fidelity of remote state preparation which provides an operational meaning to GMQD.

Because of the unavoidable interaction between a quantum system and its environment, another interesting area of investigation of the discord is its behaviour under different noisy environments [7–12]. It has been shown that quantum discord is more robust than entanglement for both Markovian and non-Markovian dissipative processes. In particular, the discord of a sort of Bell-diagonal

states subject to a phase damping noises was shown to exhibit a freezing phenomenon, [10] i.e., the quantum discord can be completely unaffected by decoherence for a finite period of time. Lang and Caves [13] provide a complete geometric picture for the frozen-discord phenomenon. The freezing phenomena have been found to be a robust feature of a class of models in the presence of non-dissipative decoherence [11, 16, 17]. The model of Mazzola *et al* has also been extended to local non-Markovian case [16]. The experimental demonstration of this phenomenon have been reported by Xu *et al.* [14] with optical systems and Auccaise *et al* [15] using NMR. However, previous works mainly focused on the measurement based discord and thus restricted to some special cases such as Bell-diagonal states. It has been shown that the quantum discord and GMQD do not necessarily imply the same ordering of two-qubit X-states [18–21]. More recently, Girolami and Adesso [22] and independently Batle *et al.* [23] provided a numerical evidences, from which one can infer that there exist other states violating the states ordering with quantum discords. It is thus worth investigating the freezing phenomena with GMQD. In this paper we consider the two qubits initially in arbitrary X-states in the presence of local dephasing noises, we derive the conditions for the freezing phenomenon in terms of GMQD. We also consider the non-Markovian dephasing case and find that our results also hold for the non-Markovian dephasing process. The fact that, under certain conditions, GMQD remain unchanged by the phase damping noise for some period of time may have potential applications in future quantum information tasks.

Before describing the main results of this paper, let us briefly review the method to compute GMQD. An arbitrary two-qubit state can be written in Bloch representation:

$$\rho = \frac{1}{4} \left[ I \otimes I + \sum_i^3 (x_i \sigma_i \otimes I + y_i I \otimes \sigma_i) + \sum_{i,j=1}^3 R_{ij} \sigma_i \otimes \sigma_j \right] \quad (1)$$

where  $x_i = \text{Tr}(\rho(\sigma_i \otimes I))$ ,  $y_i = \text{Tr}(\rho(I \otimes \sigma_i))$  are compo-

nents of the local Bloch vectors,  $\sigma_i, i \in \{1, 2, 3\}$  are the three Pauli matrices, and  $R_{ij} = \text{Tr} \rho(\sigma_i \otimes \sigma_j)$  are components of the correlation tensor. For two-qubit case, the zero-discord state is of the form  $\chi = p_1 |\psi_1\rangle \langle \psi_1| \otimes \rho_1 + p_2 |\psi_2\rangle \langle \psi_2| \otimes \rho_2$ , where  $\{|\psi_1\rangle, |\psi_2\rangle\}$  is a single-qubit orthonormal basis. Then a analytic expression of the GMQD is given by [5]

$$D_A^g(\rho) = \frac{1}{4} (\|x\|^2 + \|R\|^2 - k_{\max}) \quad (2)$$

where  $x = (x_1, x_2, x_3)^T$  and  $k_{\max}$  is the largest eigenvalue of matrix  $K = xx^T + RR^T$ . By introducing a matrix  $\mathcal{R}$  defined by

$$\mathcal{R} = \begin{pmatrix} 1 & y^T \\ x & R \end{pmatrix} \quad (3)$$

and  $3 \times 4$  matrix  $\mathcal{R}'$  through deleting the first row of  $\mathcal{R}$ . Then the analytical expression of GMQD can be further rewritten as [24]

$$D_A^g(\rho) = \frac{1}{4} \left[ \left( \sum_k \lambda_k^2 \right) - \max_k \lambda_k^2 \right] \quad (4)$$

where  $\lambda_k$  is the singular values of  $\mathcal{R}'$ . In the following discussions we use Eq.(4) to calculate GMQD. Suppose the initially state is prepared in arbitrary two-qubit X-states with the general form

where we have chosen the computational basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ . Eq.(5) is a 7-real parameter state with three real parameters along the main diagonal and two complex parameters at off-diagonal positions. A remarkable aspect of the X-states is that the initial X structure is preserved during the decoherence process, thus, it is convenient to get analytical result. At present, there are no general analytical expressions to compute measurement-based quantum discord. To get analytical results generally, we use GMQD to quantify the quantum correlation contained in the X-states and our results are summarized as the following theorem:

**Theorem:** Consider arbitrary X-states defined in Eq.(5), with each qubit being subject to the local phase damping noises. The necessary and sufficient conditions for the freezing phenomena of GMQD satisfies:

$$\begin{aligned} |\rho_{14}|^2 + |\rho_{23}|^2 &= 2\sqrt{\rho_{14}\rho_{41}\rho_{23}\rho_{32}}, \\ 8|\rho_{14}\rho_{23}| &> (\rho_{11} - \rho_{33})^2 + (\rho_{22} - \rho_{44})^2 \end{aligned} \quad (6)$$

**Proof.** First we notice that the X-states defined in Eq.(5) can be rewritten in the Bloch representation with the correlation matrix R is given by

$$R = \begin{pmatrix} \rho_{14} + \rho_{23} + \rho_{32} + \rho_{41} & i(\rho_{14} - \rho_{23} + \rho_{32} - \rho_{41}) & 0 \\ i(\rho_{14} + \rho_{23} - \rho_{32} - \rho_{41}) & -\rho_{14} + \rho_{23} + \rho_{32} - \rho_{41} & 0 \\ 0 & 0 & \rho_{11} - \rho_{22} - \rho_{33} + \rho_{44} \end{pmatrix} \quad (7)$$

and  $x = (0, 0, \rho_{11} + \rho_{22} - \rho_{33} - \rho_{44})^T$ . Thus the GMQD can be calculated according to Eq.(4) which is given by  $D_A^g(\rho) = \frac{1}{4} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \max\{\lambda_1^2, \lambda_2^2, \lambda_3^2\})$ , where

$$\begin{aligned} \lambda_1^2 &= 4(\rho_{14}\rho_{41} + \rho_{23}\rho_{32}) + 8\sqrt{\rho_{14}\rho_{41}\rho_{23}\rho_{32}} \\ \lambda_2^2 &= 4(\rho_{14}\rho_{41} + \rho_{23}\rho_{32}) - 8\sqrt{\rho_{14}\rho_{41}\rho_{23}\rho_{32}} \\ \lambda_3^2 &= 2[(\rho_{11} - \rho_{33})^2 + (\rho_{22} - \rho_{44})^2] \end{aligned} \quad (8)$$

We suppose the X-states is subject to two local Markovian phase damping noises which is given via its Kraus representation as  $\mathfrak{S}(\rho) = E_1 \rho E_1^\dagger + E_2 \rho E_2^\dagger$ , with  $E_1 = \sqrt{1-p}I$ , and  $E_2 = \sqrt{\frac{p}{2}}(|0\rangle\langle 0| - |1\rangle\langle 1|)$ . The parameter  $p$  range from 0 to 1. The phase damping noises will affect the off-diagonal elements while the diagonal elements remain unchanged. In order to derive the necessary and

sufficient conditions for when GMQD are unaffected for a finite period it suffices to consider two cases:

Case 1. If  $\rho_{14}\rho_{23} < 0$ , then  $\lambda_2^2 > \lambda_1^2$ , in order to exhibit a freezing phenomenon, we have  $\lambda_1^2 = 0$ , and  $\lambda_2^2 > \lambda_3^2$ . Thus  $D_A^g(\rho) = \frac{1}{4}\lambda_3^2$ .

Case 2. If  $\rho_{14}\rho_{23} > 0$ , in this case, we have  $\lambda_1^2 > \lambda_2^2$ , it is directly to see that the condition of GMQD remain unaffected is given by  $\lambda_2^2 = 0$  and  $\lambda_1^2 > \lambda_3^2$ .

Combined with these two conditions, the theorem is proved. ■

In order to show the application of our theorem we consider a subclass of two qubit X-states given by

$$\chi = \frac{1}{4} \left[ I \otimes I + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i \right]$$

where we choose the Bloch vectors are  $z$  directional with  $\mathbf{r} = (0, 0, r)$ ,  $\mathbf{s} = (0, 0, s)$ . According to Eq.(6), the condition becomes  $c_1^2 = 0$ ,  $c_2^2 > r^2 + c_3^2$  or  $c_2^2 = 0$ ,  $c_1^2 > r^2 + c_3^2$

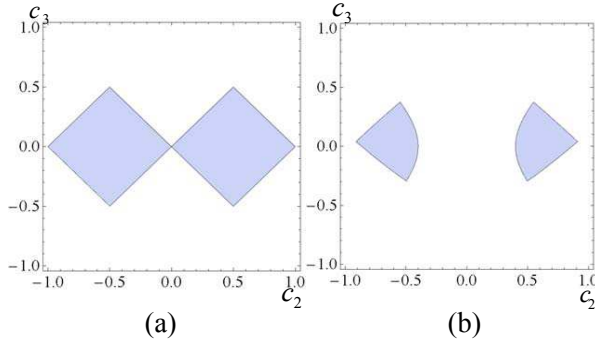


FIG. 1: (Color online). The geometry of the set of valid states satisfied Eq.(6) with different  $r$  and  $s$ , respectively. (a)  $r=s=0$ , (b)  $r=0.4, s=0.1$

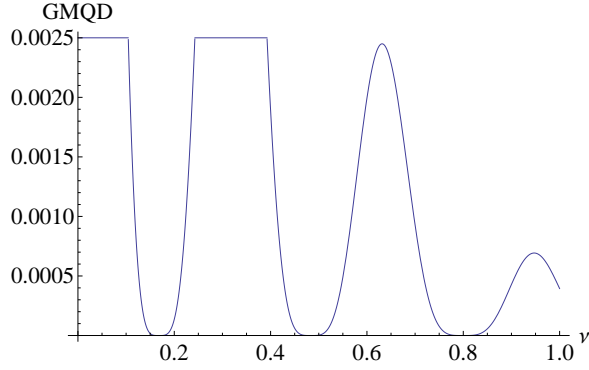


FIG. 2: (Color online). The dynamics of GMQD of Bell-diagonal states interacting with non-Markovian dephasing channels, where we have chosen  $c_1 = 0, c_2 = 0.35, c_3 = 0.1, \nu = \frac{t}{2\tau}$  with  $\tau = 2.5s$ , with  $a = 1s^{-1}$ .

which reduce to the results presented in Ref.[25]. For fixed parameters  $r$  and  $s$ , the above inequalities becomes a two-parameter set, whose geometry can be depicted. Combined with the positively condition of the eigenvalues of the density matrix  $\chi$  we can depicted the possible region of the mixed state. In Fig.1 we plot the physical region with different  $r$  and  $s$ , respectively. If  $r = s = 0$ , the mixed state reduces to the two qubit Bell-diagonal states.

Further we find that our results also holds for the non-Markovian dephasing process. The reason is that the diagonal elements of the X-states remain unchanged in the non-Markovian case, thus the proof does not altered. For concreteness, let us consider Bell-diagonal state  $\rho_{AB} = \frac{1}{4} \left( I_{AB} + \sum_{i=1}^3 c_i \sigma_i^A \sigma_i^B \right)$  subject to the non-Markovian model presented by Daffer *et al* [26], where  $\sigma_i^{A(B)}$  de-

note the Pauli operators in direction  $i$  acting on  $A(B)$ . It is shown that the map has a Kraus decomposition  $\Phi_t(\rho) = \sum_k A_k^\dagger \rho A_k$ , with  $A_i = \sqrt{1 - \Lambda(\nu)/2} \sigma_i$ ,  $A_j = 0$ ,  $A_k = 0$ ,  $A_4 = \sqrt{1 + \Lambda(\nu)/2} I$ , where  $\Lambda(\nu) = e^{-\nu} \left[ \cos(\mu\nu) + \frac{\sin(\mu\nu)}{\mu} \right]$ ,  $\mu = \sqrt{(4a\tau)^2 - 1}$ ,  $\nu = \frac{t}{2\tau}$  is the dimensionless time,  $i = 1, 2, 3$  denote the direction of noise, and  $j$  and  $k$  denotes the directions in which there is no noise. By changing the direction of the noise we can get a colored noise bit flip, bit-phase flip or phase flip channel, respectively. This process is non-Markovian according to the measure of non-Markovianity of Ref.[27]. Using the above Kraus decomposition we obtain the parameters of  $c_i$  of the Bell-diagonal states evolve as:  $c_3(t) = c_3, c_1(t) = c_1 \Lambda(\nu)^2, c_2(t) = c_2 \Lambda(\nu)^2$ , where we have chosen the phase flip channel(which means  $i = 3$ ) case and similar results can be found for bit flip and bit phase flip channel. For fixed  $c_i$  we plot the dynamic behavior of GMQD in Fig.2. It is shown that the frozen phenomenon appear two times in contrast to Markovian case where the frozen phenomenon appears only one time.

In summary, we have investigate the dynamics of GMQD for X-states under the local dephasing noises. We derive the necessary and sufficient conditions for freezing phenomenon in terms of GMQD. We also consider the non-Markovian dephasing case and find that our results can also be applied to the non-Markovian dephasing process. An open question is whether there exist other class of mixed state exhibit the frozen phenomenon. The future study involve exploring potential applications to exploit the class of initial states exhibiting frozen phenomenon in future quantum information tasks without any disturbance from the noisy environment.

This work was supported by National Natural Science Foundation of China under Grant No.10905024, No.11005029, No.11104057, the Key Project of Chinese Ministry of Education under Grant No.211080, No.210092, and the Key Program of the Education Department of Anhui Province under Grant No.KJ2011A243, No.KJ2010A287, the China Postdoctoral Science Foundation under Grant No. 20110490825, No.2011M501373, the Anhui Provincial Natural Science Foundation under Grant No.11040606M16, No.10040606Q51, the Doctoral Startup Foundation of Hefei Normal University under Grant No.2011rcjj03.

Note added. After completing this manuscript, we became aware of an interesting related works by Bo You and Li-Xiang Cen [28] recently.

[1] A. Datta, A. Shaji, and C. M. Caves, Phys. Rev. Lett. 100, 050502 (2008).

[2] K. Modi, A. Brodutch, H. Cable, T. Paterek, V. Vedral, arXiv:1112.6238 (2011).

- [3] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. 88, 017901 (2001).
- [4] L. Henderson and V. Vedral, J. Phys. A 34, 6899 (2001).
- [5] B. Dakic, V. Vedral, and C. Brukner, Phys. Rev. Lett. 105, 190502 (2010).
- [6] B. Dakic, Y. Ole Lipp, X. Ma, M. Ringbauer, S. Kropatschek, S. Barz, T. Paterek, V. Vedral, A. Zeilinger, C. Brukner, P. Walther, , arXiv:1203.1629 (2012).
- [7] T. Werlang, S. Souza, F. F. Fanchini, and C. J. Villas Boas, Phys. Rev. A 80, 024103 (2009).
- [8] J. Maziero, L. C. Celeri, R. M. Serra, and V. Vedral, Phys. Rev. A 80, 044102 (2009).
- [9] B. Wang, Z.-Y. Xu, Z.-Q. Chen, and M. Feng, Phys. Rev. A 81, 014101 (2010).
- [10] L. Mazzola, J. Piilo and S. Maniscalco, Phys. Rev. Lett. 104, 200401 (2010).
- [11] B. Li, Z.-X. Wang and S.-M. Fei, Phys. Rev. A 83, 022321(2011).
- [12] F. F. Fanchini, T. Werlang, C. A. Brasil, L. G. E. Arruda, A. O. Caldeira, Phys. Rev. A 81, 052107 (2010).
- [13] M. D. Lang, and C. M. Caves, Phys. Rev. Lett. 105, 150501 (2010).
- [14] J.-S. Xu, X.-Y. Xu, C.-F. Li, C.-J. Zhang, X.-B. Zou, and G.-C. Guo, Nat. Commun. 1, 7 (2010).
- [15] R. Auccaise, L. C. Celeri, D. O. Soares-Pinto, E. R. deAzevedo, J. Maziero, A. M. Souza, T. J. Bonagamba, R. S. Sarthour, I. S. Oliveira, R. M. Serra, Phys. Rev. Lett. 107, 140403 (2011).
- [16] L. Mazzola, J. Piilo, S. Maniscalco, International Journal of Quantum Information 9, 981 (2011).
- [17] G. Karpat and Z. Gedik, Physics Letters A 375, 4166 (2011).
- [18] F. Altintas, Optics Communications 283, 5264 (2010).
- [19] Z. Xu, W. Yang, X. Xiao, M. Feng, J. Phys. A: Math. Theor. 44, 395304 (2011).
- [20] M. Okrasa, Z. Walczak, arXiv:1109.4132 (2011).
- [21] Y. Yeo, J.-H. An, and C. H. Oh, Phys. Rev. A 82, 032340 (2010).
- [22] D. Girolami and G. Adesso, Phys. Rev. A 83, 052108 (2011).
- [23] J. Batle, A. Plastino, A. R. Plastino, and M. Casas, arXiv:1103.0704 (2011).
- [24] X.-M. Lu, Z.-J. Xi, Z. Sun, X. Wang, Quantum Inf. Comp, 10, 994 (2010).
- [25] Wei Song, Long-Bao Yu, Ping Dong, Da-Chuang Li, Ming Yang, Zhuo-Liang Cao, arXiv:1112.4318 (2011).
- [26] S. Daffer, K. Wodkiewicz, J. D. Cresser, J. K. McIver, Phys. Rev. A 70, 010304 (2004).
- [27] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. 103, 210401 (2009).
- [28] Bo You, Li-Xiang Cen, arXiv:1203.0650 (2012).